Tentamen Kansrekening – Open book

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1. Denote by X the number of fixed points of a random permutation of n objects.

   Compute $E(X)$, the variance, and $E(X^3)$!

   Hint: Use computations with indicator functions!

   Solution: Write $X = \sum_{i=1}^{n} 1_{A_i}$ where A_i is the event that i is a fixed point.

   We have for the probabilities

   $$\begin{aligned} P(A_1) &= \frac{1}{n} \\ P(A_1 \cap A_2) &= \frac{1}{n(n-1)} \\ P(A_1 \cap A_2 \cap A_3) &= \frac{1}{n(n-1)(n-2)} \end{aligned}$$

   So we get for the moments

   $$E(X) = nE(1_{A_1}) = 1$$

   and

   $$E(X^2) = (n^2 - n)P(A_1 \cap A_2) + nP(A_1) = 2$$

   So the variance is 1. Finally,

   $$\begin{aligned} E(X^3) &= n(n-1)(n-2)P(A_1 \cap A_2 \cap A_3) + 3n(n-1)P(A_1 \cap A_2) + nP(A_1) \\ &= 1 + 3 + 1 = 5 \end{aligned}$$

   Note the $3n(n-1)$ in the second term on the r.h.s. We can check the prefactors when we realize that $n(n-1)(n-2) + 3n(n-1) + n = n^3$. This is clear because they account for all possible choices of three indices between 1 and n.

2. Suppose that A and B are two events with $P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{3}$. Show that always the inequality $\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}$ holds

   Solution: The r.h.s. is clear by considering B. The l.h.s. can be seen by

   $$P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1$$

   substituting the explicit numbers.

3. Be N a random variable that is distributed according to Poisson with parameter λ.

   a) Compute the exponential moment generating function $E(e^{tN})$.

   Solution: well known from lecture and book, $E(e^{tN}) = \exp\lambda(e^t - 1)$

   b) Now take a sequence of i.i.d. random Normal variables with expected value μ and variance σ^2 and show that $E(e^{tX_i}) = e^{\frac{t^2\sigma^2}{2}+t\mu}$

   Next we define the random variable $S := \sum_{i=1}^{N} X_i$, where the random variables N and X_i are as above.

   c) Compute the exponential moment generating function $E(e^{tS})$!

   Solution:

$$
\begin{aligned}
E(e^{tN}) &= \sum_{n=0}^{\infty} P(N = n)E(e^{t\sum_{i=1}^{n} X_i}) \\
&= e^{-\lambda}\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}E(e^{tX_1})^n \\
&= \exp\lambda(e^{\frac{t^2\sigma^2}{2}+t\mu} - 1)
\end{aligned}
$$

   d) Compute $E(S)$ and the variance of S!

   Hint for d): Use c)! However, if you don't like c), you can also avoid it in the computation

   Solution: $E(S) = \lambda\mu$

   $E(S^2) = \lambda^2\mu^2 + \lambda(\sigma^2 + \mu^2)$

   $E(S^2) - E(S)^2 = \lambda(\sigma^2 + \mu^2)$

   One can avoid part c) in the computation, if you use the formula for the expected value and variance of a sum with a random number of terms that was treated in the exercises.

4. Be X an exponential random variable with parameter 1.

   (a) Compute the density of $\log X$

   Solution: For the distribution function we have $P(\log X \leq x) = 1 - e^{-e^x}$.

   Taking derivatives gives the density $e^{-e^x}e^x$.

   Be Y another exponential random variable with parameter 1, independent of X

(b) Compute the joint density of the random vector $(X, \frac{Y}{X+Y})$.

Hint: Use the multidimensional transformation formula, and note that $\frac{Y}{X+Y}$ takes values only in the interval $[0, 1]$

Solution: Put $u = x$ and $v = \frac{y}{x+y}$. Then the inverse transformation is given by $x = u$ and $y = \frac{vu}{1-v}$. The functional determinant is given by $\frac{u}{(1-v)^2}$.

So the joint density is

$$\exp(-\frac{u}{1-v})\frac{u}{(1-v)^2}$$

for $0 < u < \infty$ and $0 < v < 1$.

(c) Compute the density of $\frac{Y}{X+Y}$.

Solution: as we see from (b) by integration over u it is uniform on the interval $[0, 1]$. More precisely, the marginal distribution on the v-variable is obtained by integration over u. Making a substitution $\tilde{u} = \frac{u}{1-v}$ this integral becomes

$$\int_0^\infty \exp(-\frac{u}{1-v})\frac{u}{(1-v)^2}du = \int_0^\infty \exp(-\tilde{u})\tilde{u}d\tilde{u} = 1$$

That the last integral is 1 can be seen without looking at computations or tables from the fact that it has become v-independent because the remaining quantity must be a probability density on $[0, 1]$.

5. Be X_n a sequence of independent geometric random variables with n-dependent parameter $p_n = 1 - e^{-n}$. (We use notations according to the table in KaDe page 251, meetkundig)

   Use the Borel-Cantelli Lemma to show that X_n converges to 0 almost surely! Is the assumption of independence really necessary to reach this conclusion?

   Solution: That X_n converges to zero almost surely means equivalently that only finitely many non-zeros occur. Now, this is implied by by the convergence of the series

$$\sum_{n=1}^{\infty} P(X_n \neq 0) = \sum_{n=1}^{\infty}(1 - p_n) = \sum_{n=1}^{\infty} e^{-n} < \infty$$

   For this conclusion we don't need the independence.